THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 20
April 1, 2025 (Tuesday)

1 Recall

Recall that the **epigraph** of a function is given by $\operatorname{epi}(f) := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$. From last lesson, we say that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is **lower semi-continuous (l.s.c.)** if and only if $\operatorname{epi}(f)$ is closed. Also, if $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, then $\partial f(x) \neq \emptyset$ for all $x \in \operatorname{ri}\operatorname{dom}(f)$.

Lemma 1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex. Then*

1. *there exists $\operatorname{cl} f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{epi}(\operatorname{cl} f) = \overline{\operatorname{epi}(f)}$.*
2. $\operatorname{cl} f(x) = \liminf_{x' \to x} f(x')$ *and* $\operatorname{dom}(f) \subseteq \operatorname{dom}(\operatorname{cl} f) \subseteq \overline{\operatorname{dom}(f)}$ *and* $\operatorname{ri}\operatorname{dom}(f) = \operatorname{ri}\operatorname{dom}(\operatorname{cl} f)$.
3. $\operatorname{cl} f(x) = \sup\left\{\phi(x) : \phi(y) = A^T y + b \leq f(y),\ \forall y \in \mathbb{R}^n\right\}$.

Proof. 1. Let $\operatorname{cl} f(x) := \inf\left\{t : (x,t) \in \overline{\operatorname{epi}(f)}\right\}$ with the convention: $\inf \phi = +\infty$. Then

$$(x,t) \in \overline{\operatorname{epi}(f)} \implies \begin{cases} (x_n, t_n) \in \operatorname{epi}(f) \\ (x_n, t_n) \to (x,t) \end{cases}$$

This implies that $(x_n, t_n + c) \in \operatorname{epi}(f)$, $\forall c \geq 0$ and hence

$$(x, t+c) = \lim_{n \to +\infty} (x_n, t_n + c) \in \overline{\operatorname{epi}(f)}$$

follows from $(x,t) \in \overline{\operatorname{epi}(f)}$, so $\overline{\operatorname{epi}(f)} =: A = \operatorname{epi}(\operatorname{cl} f)$.

2. From (1), since $\overline{\operatorname{epi}(f)} = \operatorname{epi}(\operatorname{cl}(f))$ is closed, then $\operatorname{cl}(f)$ is **l.s.c** and $\operatorname{cl}(f) \leq f(x),\ \forall x \in \mathbb{R}^n$. Because we have $\operatorname{epi}(\operatorname{cl} f) \supseteq \operatorname{epi}(f)$, so it follows that $\operatorname{cl} f(x) \leq \liminf_{x' \to x} \operatorname{cl} f(x') \leq \liminf_{x' \to x} f(x')$. Moreover, it follows that
$$(x, \operatorname{cl} f(x)) \in \operatorname{epi}(\operatorname{cl} f) = \overline{\operatorname{epi}(f)}.$$
There exists $(x_n, t_n) \in \operatorname{epi}(f)$ with $(x_n, t_n) \to (x, \operatorname{cl} f(x))$ such that $f(x_n) \leq t_n$. So, it means that $\operatorname{cl} f(x) = \lim_{n \to +\infty} t_n \geq \limsup_{n \to +\infty} f(x_n) \geq \lim \inf_{x' \to x} f(x')$. Together with the above, we prove the equality that $\operatorname{cl} f(x) = \liminf_{x' \to x} f(x')$.
Now, for any $x \in \operatorname{dom}(f)$, we have $f(x) < +\infty$ and hence
$$\operatorname{cl} f(x) \leq f(x) < +\infty \implies x \in \operatorname{dom}(\operatorname{cl} f).$$
So $\operatorname{dom}(f) \subseteq \operatorname{dom}(\operatorname{cl} f)$. Also, for all $x \in \operatorname{dom}(\operatorname{cl} f)$, we have $\operatorname{cl} f(x) < +\infty$ and hence there exists sequence $(x_n, f(x_n)) \to (x, \operatorname{cl} f(x))$, $x_n \in \operatorname{dom}(f)$, so $x \in \overline{\operatorname{dom}(f)}$ and this proves the first part of subset relations.
To prove the second part, for $x \in \operatorname{Int}(\operatorname{dom}(\operatorname{cl} f))$, equivalently, we have
$$\operatorname{cl} f(y) < +\infty,\ \forall y \in B_\varepsilon(x)$$
where $B_\varepsilon(x) := \{y : \|y - x\| \leq \varepsilon\}$. Then, because $\operatorname{dom}(f)$ is dense in $B_\varepsilon(x)$ and $\operatorname{dom}(f)$ is convex, so $\operatorname{int}(B_\varepsilon(x)) \subseteq \operatorname{dom}(f)$ and hence $x \in \operatorname{int}(\operatorname{dom}(f))$.

3. Define $g(x) = \sup\left\{\phi(x) : \phi(y) = A^T y + b \leq f(y),\ \forall y \in \mathbb{R}^n\right\}$.

   - For all continuous ϕ, as supremum of continuous functions, so g is **l.s.c.**.
   - For all convex ϕ, we have g is **convex**.

   Then, for all $x \in \operatorname{ri}(\operatorname{dom}(f))$, we have $g(x) \leq f(x)$, so there exists $v \in \partial f(x)$, or equivalently

$$\begin{aligned} f(y) &\geq f(x) + v^T(y-x), \quad \forall y \in \mathbb{R}^n \\ &= v^T y + \big(f(x) - v^T x\big) \\ &= \phi(y),\ \forall y \in \mathbb{R}^n \end{aligned}$$

   and $f(x) = \phi(x)$, so $g(x) = \sup\left\{\phi(x) : \phi(y) = A^T y + b \leq f(y),\ \forall y \in \mathbb{R}^n\right\} \geq f(x)$, together with the above, we get $f(x) = g(x) = \operatorname{cl} f(x)$.
   Now, we have

   - $\operatorname{cl} f(x)$ is convex and **l.s.c.**
   - g is convex and **l.s.c.**
   - $\operatorname{cl}(f) = g$ on $\operatorname{ri}(\operatorname{dom}(f))$.

   So, we have $\operatorname{cl}(f) = g$ on relative boundary of $\operatorname{dom}(f)$.
   Now, to complete the proof, we need to handle the case of $\mathbb{R}^n \setminus \overline{\operatorname{dom}(f)}$.
   For all $x \notin \overline{\operatorname{dom}(f)}$, then

$$f(y) = +\infty,\ \ \forall y \in B_\varepsilon(x) \quad \text{and} \quad \operatorname{cl} f(y) = +\infty$$

   so for all t, there exists $\phi(y) = A^T y + b \leq f(y)$ and $\phi(x) \geq t$. Then, we can deduce that

$$g(x) = \sup\left\{\phi(x) : \phi(y) = A^T y + b \leq f(y),\ \forall y \in \mathbb{R}^n\right\} = +\infty = \operatorname{cl} f(y)$$

□

2 Conjugate Function

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$. We define its **conjugate** by

$$f^*(d) := \sup_{x \in \mathbb{R}^n} \big(d^T x - f(x)\big) = \sup\left\{\phi_x(d) : \phi_x(v) = v^T x - f(x),\ \forall v \in \mathbb{R}^n\right\}$$

Also, we define the **bi-conjugate** of f as follows:

$$\left(f^*\right)^*(x) := \sup_{d \in \mathbb{R}^n} \big(x^T d - f^*(d)\big)$$

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, then $(f^*)^* = \operatorname{cl} f$.*

Proof. By definition, we have

$$(f^*)^* = \sup_{d \in \mathbb{R}^n} \big(x^T d - f^*(d)\big) = \sup_{\substack{d \in \mathbb{R}^n \\ a \geq f^*(d)}} \big(x^T d - a\big).$$

Since $a \geq f^*(d)$, so it is equivalent to say that

$$a \geq f^*(d) = \sup_{x \in \mathbb{R}^n} \big(d^T x - f(x)\big)$$

so $a \geq d^T y - f(y),\ \forall y \in \mathbb{R}^n$ and hence $f(y) \geq d^T y - a,\ \forall y \in \mathbb{R}^n$. On the other hand, since

$$\sup_{\substack{d \in \mathbb{R}^n \\ a \geq f^*(d)}} \left(x^T d - a \right) = \sup_{d \in \mathbb{R}^n} \left\{ x^T d - a : f(y) \geq d^T y - a,\ \forall y \in \mathbb{R}^n \right\}$$

Now, we take $\phi(x) = x^T d - a$, then we have

$$\sup \left\{ \phi(x) : \phi(y) \text{ is affine and } \phi(y) \leq f(y),\ \forall y \in \mathbb{R}^n \right\} = \operatorname{cl} f$$

□

Remarks. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which is not necessarily convex, then

$$\begin{aligned} (f^*)^* &= \sup \left\{ \phi(x) : \phi(y) \text{ is affine function and } \phi(y) \leq f(y),\ \forall y \right\} \\ &= \sup \left\{ \phi(x) :\ \phi \text{ is convex},\ \phi(y) \leq f(y),\ \forall y \right\} \end{aligned}$$

is called the **convex envelope** (the **biggest** convex function below f) of f.

Remarks. We have the following remarks:

1. $v \mapsto \phi_x(v)$ is affine (which is continuous and convex) $\implies$ $d \mapsto f^*(d)$ is **l.s.c.** and convex $\implies$ $(f^*)^*$ is also **l.s.c.** and convex.
2. $0 \in \operatorname{dom}(f^*)$ if and only if $f^*(0) < +\infty \iff \sup\limits_{x \in \mathbb{R}^n} (-f(x)) < +\infty \iff \inf\limits_{x \in \mathbb{R}^n} f(x) > -\infty$.
3. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, i.e. $f \not\equiv +\infty$. Then $f^*(d) \in \mathbb{R} \cup \{+\infty\}$.
4. $\hat{x} \in \partial f^*(0)$ and f is convex $\iff$ $f^*(d) \geq f^*(0) + \hat{x}^T d,\ \forall d \in \mathbb{R}^n$, it implies that $\hat{x}$ is a minimizer of f.

3 Example for Conjugate Functions

Example 1. For a constant function $f(x) \equiv -a$, then to compute its conjugate function, we consider

$$\begin{aligned} f^*(d) &= \sup_x \left(d^T x - f(x) \right) = \sup_x \left(d^T x + a \right) \\ &= \begin{cases} a, & \text{if } d = 0 \\ +\infty, & \text{if } d \neq 0 \end{cases} \end{aligned}$$

Example 2. For an affine function $f(x) = A^T x + b$, then

$$\begin{aligned} f^*(d) &= \sup_{x \in \mathbb{R}^n} \left(d^T x - A^T x - b \right) \\ &= \begin{cases} -b, & \text{if } d = A \\ +\infty, & \text{if } d \neq A \end{cases} \end{aligned}$$

Example 3. Given that $f(x) = \dfrac{1}{2} x^T A x$, then

$$\begin{aligned} f^*(d) &= \sup_x \left(d^T x - \frac{1}{2} x^T A x \right) \\ &= \frac{1}{2} d^T A^{-1} d \end{aligned}$$

Example 4. For a Lagrangian function $L(x, \lambda) := f(x) + \lambda g(x)$ and $d(\lambda) := \sup\limits_{x \in \mathbb{R}^n} L(x, \lambda)$. Then

$$d(\lambda) := \sup_{x \in \mathbb{R}^n} L(x, \lambda) = \sup_{x \in \mathbb{R}^n} (\lambda x - (-f(x))) = (-f)^*(\lambda).$$

— End of Lecture 20 —